

# Nonlinear diffusion equations as asymptotic limits of Cahn–Hilliard systems

Pierluigi Colli

Dipartimento di Matematica, Università di Pavia

and IMATI C.N.R. Pavia

Via Ferrata 1, 27100 Pavia, Italy

E-mail: [pierluigi.colli@unipv.it](mailto:pierluigi.colli@unipv.it)

Takeshi Fukao

Department of Mathematics, Faculty of Education

Kyoto University of Education

1 Fujinomori, Fukakusa, Fushimi-ku, Kyoto 612-8522 Japan

E-mail: [fukao@kyokyo-u.ac.jp](mailto:fukao@kyokyo-u.ac.jp)

## Abstract

An asymptotic limit of a class of Cahn–Hilliard systems is investigated to obtain a general nonlinear diffusion equation. The target diffusion equation may reproduce a number of well-known model equations: Stefan problem, porous media equation, Hele–Shaw profile, nonlinear diffusion of singular logarithmic type, nonlinear diffusion of Penrose–Fife type, fast diffusion equation and so on. Namely, by setting the suitable potential of the Cahn–Hilliard systems, all of these problems can be obtained as limits of the Cahn–Hilliard related problems. Convergence results and error estimates are proved.

**Key words:** Cahn–Hilliard system, Stefan problem, porous media equation, Hele–Shaw profile, fast diffusion equation.

**AMS (MOS) subject classification:** 35K61, 35K25, 35B25, 35D30, 80A22.

## 1 Introduction

In this paper, we are interested in a discussion of the nonlinear diffusion problem

$$\frac{\partial u}{\partial t} - \Delta \xi = g, \quad \xi \in \beta(u) \quad \text{in } Q := \Omega \times (0, T), \quad (1.1)$$

$$\partial_\nu \xi = h \quad \text{in } \Sigma := \Gamma \times (0, T), \quad (1.2)$$

$$u(0) = u_0 \quad \text{in } \Omega, \quad (1.3)$$

as an asymptotic limit of the following Cahn–Hilliard system

$$\frac{\partial u}{\partial t} - \Delta \mu = 0 \quad \text{in } Q, \quad (1.4)$$

$$\mu = -\varepsilon \Delta u + \xi + \pi_\varepsilon(u) - f, \quad \xi \in \beta(u) \quad \text{in } Q, \quad (1.5)$$

$$\partial_\nu \mu = \partial_\nu u = 0 \quad \text{in } \Sigma, \quad (1.6)$$

$$u(0) = u_{0\varepsilon} \quad \text{in } \Omega, \quad (1.7)$$

as  $\varepsilon \searrow 0$ , where  $0 < T < +\infty$  denotes a finite time and  $\Omega \subset \mathbb{R}^d$ ,  $d = 2$  or  $3$ , is a bounded domain with smooth boundary  $\Gamma$ ; the symbol  $\Delta$  stands for the Laplacian, and  $\partial_\nu$  denotes the outward normal derivative on  $\Gamma$ . In the nonlinear diffusion term,  $\beta$  is a maximal monotone graph and  $\pi_\varepsilon$  is an anti-monotone function which tends to 0 in a suitable way as  $\varepsilon \searrow 0$ . It is well known that the Cahn–Hilliard system (1.4)–(1.7) is characterized by the nonlinear term  $\beta + \pi_\varepsilon$ , which represents some derivative (actually, a non-smooth  $\beta$  plays as the subdifferential of a proper convex and lower semicontinuous function) of a multi-well function  $W$ . Usually referred as the double well potential, we can take, as a simple example,  $W(r) = (1/4)(r^2 - \varepsilon)^2$ ; in this case, we have that  $\beta(r) = r^3$  and  $\pi_\varepsilon(r) = -\varepsilon r$  for all  $r \in \mathbb{R}$ ,  $\pi_\varepsilon$  depending on  $\varepsilon > 0$  (see the details of this prototype in [11, 21]). Actually, the Cahn–Hilliard system (1.4)–(1.7) has been treated under various frameworks for  $\beta + \pi_\varepsilon$ , in particular for graphs  $\beta$  singular with bounded domain or even non-smooth subdifferentials of bounded intervals. On the other hand, it is clear that the corresponding problem (1.1)–(1.3) may represent various kind of nonlinear diffusion problems: Stefan problem, porous media equation, Hele-Shaw profile, diffusion for a singular logarithmic potential, nonlinear diffusion of Penrose–Fife type, and fast diffusion equation (see the later Examples 1–6).

The main objective of this paper is to show, for a fixed graph  $\beta$  and for some known datum  $f$  precisely related to the data  $g$  and  $h$  in (1.1)–(1.2) (cf. (2.6)–(2.7)), the convergence of the solutions to the Cahn–Hilliard system (1.4)–(1.7) to the respective solution of the nonlinear diffusion problem (1.1)–(1.3). Namely, by performing some asymptotic limit  $\varepsilon \searrow 0$  in (1.4)–(1.7), we naturally reaffirm the existence of solutions to (1.1)–(1.3). Of course, the initial values  $u_{0\varepsilon}$  in (1.7) should suitably converge to the initial datum  $u_0$  in (1.3). By a similar procedure, the Stefan problem with dynamic boundary conditions was obtained as asymptotic limit in [22]. Moreover, as the solution to the limiting problem is also unique, we can prove an error estimate for the difference of solutions in suitable norms.

A brief outline of this paper along with a short description of the various items is as follows. In Section 2, the convergence theorem is stated. Firstly, we set the notation that is used in the paper. Next, we introduce the target problem (P) for the nonlinear diffusion equation and recall the problem  $(P)_\varepsilon$  for the Cahn–Hilliard approximating system; we also state a mathematical result for  $(P)_\varepsilon$  in Proposition 2.1. At this point, we list and deal with various examples for the problem (P). All of these problems introduced in the Examples 1–6 are included in the framework of Theorem 2.3, which are stated shortly after and are focused on the convergence of Cahn–Hilliard systems  $(P)_\varepsilon$  to the nonlinear diffusion problem (P).

In Section 3, we detail the uniform estimates that will be useful to show the convergence results. In order to guarantee enough regularity for the unknowns of  $(P)_\varepsilon$ , we consider the

regularized problem  $(P)_{\varepsilon,\lambda}$  in which  $\beta$  is replaced by its Yosida approximation  $\beta_\lambda$ ,  $\lambda > 0$ . After deriving all the estimates on  $(P)_{\varepsilon,\lambda}$ , we infer the same kind of uniform estimates and consequent regularities for  $(P)_\varepsilon$ .

In Section 4, the proof of Theorem 2.3 is given. The strategy of the proof is quite standard: by exploiting the uniform estimates, we pass to the limit as  $\varepsilon \searrow 0$ ; on the other hand, let us put some emphasis on the monotonicity argument. The uniqueness for  $(P)$  is also discussed there.

In Section 5, we prove the error estimate stated in Theorem 5.1 by applying a special bootstrap argument. In Section 6, we can improve our results enhancing the error estimate by Theorem 6.1. Actually, under a stronger assumption for the heat source  $f$ , we can neglect an already required condition for the growth of  $\beta$  and, in addition, treat a wider class of problems, in particular the ones outlined in Examples 5 and 6.

A final Section 7 contains the proof of an auxiliary proposition. A detailed index of sections and subsections is reported here.

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## 2 Convergence results

In this section, we state the main results.

### 2.1 Notation

We use the spaces  $H := L^2(\Omega)$ ,  $V := H^1(\Omega)$  with usual norms  $|\cdot|_H$ ,  $|\cdot|_V$  and inner products  $(\cdot, \cdot)_H$ ,  $(\cdot, \cdot)_V$ , respectively. Moreover, we introduce the space

$$W := \{z \in H^2(\Omega) : \partial_\nu z = 0 \text{ a.e. on } \Gamma\}.$$

The symbol  $V^*$  denotes the dual space of  $V$  and the pair  $\langle \cdot, \cdot \rangle_{V^*, V}$  stands for the duality pairing between  $V^*$  and  $V$ . Define  $m : V^* \rightarrow \mathbb{R}$  by

$$m(z^*) := \frac{1}{|\Omega|} \langle z^*, 1 \rangle_{V^*, V} \quad \text{for all } z^* \in V^*,$$

where  $|\Omega|$  denotes the volume of  $\Omega$ . If  $z^* \in H$ ,  $m(z^*)$  gives the mean value of  $z^*$ , i.e.,

$$m(z^*) = \frac{1}{|\Omega|} \int_{\Omega} z^* dx.$$

Under these setting, we prepare a linear operator  $\mathcal{N} : D(\mathcal{N}) \subseteq V^* \rightarrow V$ . Define  $D(\mathcal{N}) := \{w^* \in V^* : m(w^*) = 0\}$ ; then, for  $w^* \in D(\mathcal{N})$ , we let  $w = \mathcal{N}w^*$  if  $w \in V$ ,  $m(w) = 0$  and  $w$  is a solution of the following variational equality

$$\int_{\Omega} \nabla w \cdot \nabla z dx = \langle w^*, z \rangle_{V^*, V} \quad \text{for all } z \in V. \quad (2.1)$$

If  $w^* \in D(\mathcal{N}) \cap H$ , then it turns out that  $w (= \mathcal{N}w^*)$  solves the elliptic problem

$$\begin{cases} -\Delta w = w^* & \text{a.e. in } \Omega, \\ \partial_{\nu} w = 0 & \text{a.e. in } \Gamma, \\ m(w) = 0 \end{cases}$$

and, in particular,  $w \in W$ . We have the following property of  $\mathcal{N}$ : if  $v = \mathcal{N}v^*$  and  $w = \mathcal{N}w^*$ , then

$$\begin{aligned} \langle w^*, \mathcal{N}v^* \rangle_{V^*, V} &= \langle w^*, v \rangle_{V^*, V} = \int_{\Omega} \nabla w \cdot \nabla v dx \\ &= \langle v^*, w \rangle_{V^*, V} = \langle v^*, \mathcal{N}w^* \rangle_{V^*, V} \quad \text{for all } v^*, w^* \in D(\mathcal{N}). \end{aligned} \quad (2.2)$$

Hence, by defining

$$|w^*|_{V^*}^2 := |\nabla \mathcal{N}(w^* - m(w^*))|_{H^d}^2 + |m(w^*)|^2 \quad \text{for all } w^* \in V^*, \quad (2.3)$$

it is clear that  $|\cdot|_{V^*}$  yields a norm of  $V^*$ .

We also recall the Poincaré–Wirtinger inequality: there is a positive constant  $c_P$  such that

$$|z|_V^2 \leq c_P |\nabla z|_{H^d}^2 \quad \text{for all } z \in V \text{ with } m(z) = 0. \quad (2.4)$$

## 2.2 Solution of the Cahn–Hilliard system

In this subsection, we recall the well-known result for the solvability for the Cahn–Hilliard system.

Let us emphasize that we term (P) the target problem expressed by (1.1)–(1.3): this is an initial and non-homogeneous Neumann boundary value problem for the nonlinear diffusion equation (1.1), where  $g : Q \rightarrow \mathbb{R}$ ,  $h : \Sigma \rightarrow \mathbb{R}$  and  $u_0 : \Omega \rightarrow \mathbb{R}$ , are the given data.

Moreover, for  $\varepsilon > 0$  we let  $(P)_\varepsilon$  denote the Cahn–Hilliard initial-boundary value problem (1.4)–(1.7), in which  $f : Q \rightarrow \mathbb{R}$  and  $u_{0\varepsilon} : \Omega \rightarrow \mathbb{R}$  appear as the prescribed data and should be in some relation with  $g$ ,  $h$  and  $u_0$ , as well as  $\pi_\varepsilon$  has to enjoy some properties for small  $\varepsilon$ .

A simple remark concerning  $(P)_\varepsilon$  is that, as it is usual for Cahn–Hilliard systems, equation (1.4) and the second boundary condition in (1.6) imply consevation of the mean value for  $u$ , that is,  $m(u(t)) = m(u_{0\varepsilon})$  for all  $t > 0$ . Indeed it suffices to integrate (1.4) by parts in space and time using (1.6) and (1.7). At the same time, we are interested to set the same condition of mass (or mean value) conservation for the solutions to  $(P)$ .

Thus, we prescribe the data  $g$  and  $h$  such that

$$\int_{\Omega} g(t)dx + \int_{\Gamma} h(t)d\Gamma = 0 \quad \text{for a.a. } t \in (0, T); \quad (2.5)$$

then, by simply integrating (1.1) over  $\Omega$  and using (1.2), we find that  $(d/dt) \int_{\Omega} u(t)dx = 0$ , whence

$$\frac{1}{|\Omega|} \int_{\Omega} u(t)dx = \frac{1}{|\Omega|} \int_{\Omega} u_0dx = m(u_0) =: m_0$$

for all  $t \in [0, T]$ . Then, we can specify  $f$  acting in (1.5) as an arbitrary solution of the following elliptic problem:

$$\begin{cases} -\Delta f(t) = g(t) & \text{a.e. in } \Omega, \\ \partial_{\nu} f(t) = h(t) & \text{in the sense of traces on } \Gamma, \end{cases} \quad (2.6)$$

for a.a.  $t \in (0, T)$ , that is,

$$\int_{\Omega} \nabla f(t) \cdot \nabla z dx = \int_{\Omega} g(t)z dx + \int_{\Gamma} h(t)z_{\Gamma} d\Gamma \quad \text{for all } z \in V, \quad (2.7)$$

where  $z_{\Gamma}$  denotes the trace of  $z$  on  $\Gamma$ . Throughout this paper, we assume that

(A1)  $\beta$  is a maximal monotone graph in  $\mathbb{R} \times \mathbb{R}$  with effective domain  $D(\beta)$  such that  $\text{int } D(\beta) \neq \emptyset$ , and  $\beta$  is the subdifferential

$$\beta = \partial \widehat{\beta}$$

of some convex and lower semicontinuous function  $\widehat{\beta} : \mathbb{R} \rightarrow [0, +\infty]$  satisfying  $\widehat{\beta}(0) = 0$ . This entails that  $0 \in \beta(0)$ ;

(A2) there exist two positive constants  $c_1, c_2$  such that

$$\widehat{\beta}(r) \geq c_1 |r|^2 - c_2 \quad \text{for all } r \in \mathbb{R};$$

(A3)  $\pi_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$  is a Lipschitz continuous function for all  $\varepsilon \in (0, 1]$ . Moreover, there exist a positive constant  $c_3$  and a strictly increasing continuous function  $\sigma : [0, 1] \rightarrow [0, 1]$  such that  $\sigma(0) = 0$ ,  $\sigma(1) = 1$  and

$$|\pi_\varepsilon(0)| + |\pi'_\varepsilon|_{L^\infty(\mathbb{R})} \leq c_3 \sigma(\varepsilon) \quad \text{for all } \varepsilon \in (0, 1]; \quad (2.8)$$

(A4)  $g \in L^2(0, T; H)$ ,  $h \in L^2(0, T; L^2(\Gamma))$  and  $g, h$  satisfy (2.5). Then, we can fix a solution  $f \in L^2(0, T; V)$  of (2.7);

(A5)  $u_0 \in H$  with  $\widehat{\beta}(u_0) \in L^1(\Omega)$  and  $m_0 \in \text{int } D(\beta)$ . Moreover, let  $u_{0\varepsilon} \in V$  fulfill  $m(u_{0\varepsilon}) = m_0$  and

$$|u_{0\varepsilon}|_H^2 \leq c_4, \quad \int_{\Omega} \widehat{\beta}(u_{0\varepsilon}) dx \leq c_4, \quad \varepsilon |\nabla u_{0\varepsilon}|_{H^d}^2 \leq c_4 \quad (2.9)$$

for some positive constant  $c_4$  and for all  $\varepsilon \in (0, 1]$ ; in addition,  $u_{0\varepsilon} \rightarrow u$  strongly in  $H$  as  $\varepsilon \searrow 0$ .

The existence of a family of data  $\{u_{0\varepsilon}\}$  satisfying (A5) is checked in the Appendix.

In order to make clear the generality of our setting, we give here some examples in which assumptions (A1)–(A3) hold.

**Example 1 [*Stefan problem*].** The Stefan problem is a well-known model for the mathematical description of the solid-liquid phase transition. A number of results is available in the literature for the Stefan problem, let us just quote e.g. [19, 24]. Using the weak formulation for this sharp interface model, as in [19] one can take  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  a piecewise linear function and  $\pi_\varepsilon$  as follows:

$$\beta(r) = \begin{cases} k_s r & \text{if } r < 0, \\ 0 & \text{if } 0 \leq r \leq L, \\ k_\ell(r - L) & \text{if } r > L, \end{cases} \quad \pi_\varepsilon(r) = \begin{cases} \varepsilon \frac{L}{2} & \text{if } r < 0, \\ \varepsilon \left( \frac{L}{2} - r \right) & \text{if } 0 \leq r \leq L, \\ -\frac{\varepsilon}{2} L & \text{if } r > L \end{cases}$$

for all  $r \in \mathbb{R}$ , where  $k_s, k_\ell > 0$  stand for the heat conductivities on the solid and liquid region, respectively;  $L > 0$  is the latent heat coefficient. In this model,  $u$  and  $\beta(u)$  represent the enthalpy and the temperature, respectively. One can see [7, 22, 27] and references therein about the Stefan problem and its abstract framework.

**Example 2 [*Porous media equation*].** Let us consider the dynamics of a gas in a porous medium. Let the unknown parameter  $u$  be its density. The dynamics (1.1)–(1.3) is suited for this case within the following setting of  $\beta$  and corresponding  $\pi_\varepsilon$ :

$$\beta(r) = |r|^{q-1} r, \quad \pi_\varepsilon(r) = -\varepsilon r \quad \text{for } r \in \mathbb{R},$$

with the exponent  $q > 1$ . About porous media equation, there is a large amount of related work, for example, [1–3, 6, 25, 29–31, 38, 44] and [43] so to quote a list of papers and a monograph.

**Example 3 [*Hele-Shaw profile*].** The Hele-Shaw profile is characterized by the limiting behaviour with respect to the Stefan problem as the conductivities blow up. For simplicity, we can take  $\beta$  as the inverse of the Heaviside graph  $\mathcal{H}$

$$\mathcal{H}(r) := \begin{cases} 0 & \text{if } r < 0, \\ [0, 1] & \text{if } r = 0, \\ 1 & \text{if } r > 0 \end{cases} \quad \text{for } r \in \mathbb{R}, \quad (2.10)$$

that is,

$$\beta(r) = \mathcal{H}^{-1}(r) = \partial I_{[0,1]}(r) \quad \text{for } r \in [0, 1], \quad \pi_\varepsilon(r) = \varepsilon \pi(r) \quad \text{for } r \in \mathbb{R},$$

where  $\partial I_{[0,1]}$  is the subdifferential of the indicator function  $I_{[0,1]}$  of the interval  $[0, 1]$ . The corresponding  $\pi_\varepsilon$  is defined in terms of some  $C^1$  function  $\pi$ , strictly decreasing and vanishing at  $r = 1/2$ . More details and other references can be found in [2, 3, 31, 35, 39].

**Example 4 [*Nonlinear diffusion with singular logarithmic law*].** The double well potential is chosen in order that its derivative is of singular logarithmic type. In this case,  $\beta$  is defined in an open interval, for instance  $(-1, 1)$ , and it becomes singular when it approaches  $-1$  and  $1$ . We can take

$$\beta(r) = |r| \ln \frac{1+r}{1-r} \quad \text{for } r \in (-1, 1), \quad \pi_\varepsilon(r) = -\varepsilon \alpha r \quad \text{for } r \in \mathbb{R},$$

for a fixed positive constant  $\alpha$ . Please note that the domain of  $\hat{\beta}$  is the closed interval  $[-1, 1]$ . The double well structure is reproduced also in this case. Logarithmic nonlinearities can be found in a number of contributions devoted to the Cahn–Hilliard systems (see the recent contributions [9, 10, 12, 13] and references therein).

The assumption (A2) plays a role for ensuring the existence of a solution to the limit problem, cf. [32]. Actually, we can avoid it provided we replace (A4) with the stronger regularity assumption (cf. (2.5))

(A6)  $g \in L^2(0, T; H)$ ,  $h = 0$  a.e. on  $\Sigma$ ,  $m(g(t)) = 0$  for a.a.  $t \in (0, T)$ . Then, let  $f \in L^2(0, T; W)$  satisfy

$$\int_{\Omega} \nabla f(t) \cdot \nabla z dx = \int_{\Omega} g(t) z dx \quad \text{for all } z \in V. \quad (2.11)$$

Indeed, in view of (2.6), note that  $f$  fulfils the Neumann homogeneous boundary condition and  $\Delta f$  is bounded in  $L^2(0, T; L^2(\Omega))$ , whence  $f \in L^2(0, T; W)$  by elliptic regularity. Assumption (A6) will be especially useful in Section 6 to improve the error estimate. On the other hand, the convergence result stated in Theorem 2.3 ensures the existence of a solution to the limit problem (P) as well. In this respect, our Theorem 2.3 turns out a generalization of [32].

Here, we give two additional examples fitting our framework in the case when (A2) does not hold.

**Example 5** [*Nonlinear diffusion of Penrose–Fife type*]. We take a variation of the Stefan problem, written as

$$\frac{\partial}{\partial t}(\theta + L\mathcal{H}(\theta - \theta_c)) - \Delta \left( -\frac{1}{\theta} \right) = g \quad \text{in } Q, \quad (2.12)$$

where  $\theta > 0$  denotes the absolute temperature,  $\theta_c > 0$  is a critical temperature around which the phase change occurs, and the graph  $\mathcal{H}$  is the same as in (2.10). If  $v$  is the selection from  $\zeta(\theta) := \theta + L\mathcal{H}(\theta - \theta_c)$ , then we can rewrite (2.12) as

$$\frac{\partial v}{\partial t} - \Delta \gamma(v) = g \quad \text{in } Q,$$

where  $\gamma$  is the composition of  $\theta \mapsto -1/\theta$  and the inverse graph of  $\zeta$ . Since  $\gamma$  does not go across the origin, we change the variable and set

$$u = v - \theta_c, \quad \beta(u) = \gamma(u + \theta_c) + \frac{1}{\theta_c},$$

in order to match the assumption (A1). Note that in this case,  $\pi_\varepsilon$  can be taken exactly as in the Example 1 of the Stefan problem, while the assumption (A2) is not satisfied due to the behavior of  $\beta$  as  $r \rightarrow +\infty$ . The limiting problem and variations of it were discussed in [14–17, 23, 37].

**Example 6** [*Fast diffusion equation*]. This setting is similar to the one of the porous media equation

$$\beta(r) = |r|^{q-1}r, \quad \pi_\varepsilon(r) = -\varepsilon r, \quad r \in \mathbb{R},$$

but with  $0 < q < 1$ , so that there should be extinction of the solution in a finite time (see, e.g., [5, 6, 20, 25, 28, 40, 41, 43]). The extreme cases for  $q$  are  $q = 1$ , which corresponds to the *linear* heat equation, and  $q = 0$ , which yields

$$\beta(r) = \begin{cases} -1 & \text{if } r < 0, \\ [-1, 1] & \text{if } r = 0, \\ 1 & \text{if } r > 0, \end{cases} \quad \pi_\varepsilon(r) = -\varepsilon r, \quad r \in \mathbb{R},$$

that is, a sign graph with similar behavior as  $\mathcal{H}$  in (2.10). Note that whenever  $0 \leq q < 1$ ,  $\pi_\varepsilon$  turns out to act as a perturbation at infinity: outside a bounded interval (whose length depends on  $\varepsilon > 0$ ) the potential  $\widehat{\beta} + \widehat{\pi}_\varepsilon$  becomes negative. In all of these cases, 0 is a local minimum and there are two absolute symmetric maxima. Also in this situation, assumption (A2) does not hold.

Now, let us go back to our general theory and formulate an existence and uniqueness result for the problem  $(P)_\varepsilon$  (see, e.g., [9, 18, 34, 36]).



**Proposition 2.1.** *Assume either (A1)–(A5) or (A1), (A3) with  $\sigma(\varepsilon) = \varepsilon^{1/2}$ , (A5) and (A6). Then, for every  $\varepsilon \in (0, 1]$  there exists a triplet  $(u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon)$  with*

$$\begin{aligned} u_\varepsilon &\in H^1(0, T; V^*) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \\ \mu_\varepsilon &\in L^2(0, T; V), \quad \xi_\varepsilon \in L^2(0, T; H), \end{aligned}$$

*satisfying*

$$\langle u'_\varepsilon(t), z \rangle_{V^*, V} + \int_\Omega \nabla \mu_\varepsilon(t) \cdot \nabla z dx = 0 \quad \text{for all } z \in V, \quad (2.13)$$

$$\mu_\varepsilon(t) = -\varepsilon \Delta u_\varepsilon(t) + \xi_\varepsilon(t) + \pi_\varepsilon(u_\varepsilon(t)) - f(t) \quad \text{in } H, \quad (2.14)$$

*for a.a.  $t \in (0, T)$ , and*

$$\xi_\varepsilon \in \beta(u_\varepsilon) \quad \text{a.e. in } Q, \quad (2.15)$$

$$u_\varepsilon(0) = u_{0\varepsilon} \quad \text{a.e. in } \Omega. \quad (2.16)$$

*Moreover, there exists a positive constant  $M$ , independent of  $\varepsilon > 0$ , such that*

$$\int_0^t |u'_\varepsilon(s)|_{V^*}^2 ds + \varepsilon |\nabla u_\varepsilon(t)|_{H^d}^2 + |u_\varepsilon(t)|_H^2 \leq M, \quad (2.17)$$

$$\int_0^t |\mu_\varepsilon(s)|_V^2 ds + \int_0^t |\xi_\varepsilon(s)|_H^2 ds + \int_0^t |\varepsilon u_\varepsilon(s)|_W^2 ds \leq M, \quad (2.18)$$

*for all  $t \in [0, T]$ .*

Since this type of the problem has been treated in other papers, we only sketch the key points of the proof, in particular estimates (2.17)–(2.18), in the next section.

## 2.3 Convergence theorem

In this subsection, we define the weak solution for the nonlinear diffusion problem (P). Then, we state the convergence results.

**Definition 2.2.** *A triplet  $(u, \mu, \xi)$  with*

$$u \in H^1(0, T; V^*) \cap L^\infty(0, T; H), \quad \mu, \xi \in L^2(0, T; V)$$

*is called weak solution of (P) if  $u, \mu, \xi$  satisfy*

$$\langle u'(t), z \rangle_{V^*, V} + \int_\Omega \nabla \mu(t) \cdot \nabla z dx = 0 \quad \text{for all } z \in V \text{ and a.a. } t \in (0, T), \quad (2.19)$$

$$\mu = \xi - f, \quad \xi \in \beta(u) \quad \text{a.e. in } Q, \quad (2.20)$$

$$u(0) = u_0 \quad \text{a.e. in } \Omega. \quad (2.21)$$

Our first result is related to the convergence of the solution of the Cahn–Hilliard system  $(P)_\varepsilon$  to the weak solution of the nonlinear diffusion equation (P).

**Theorem 2.3.** *Assume either (A1)–(A5) or (A1), (A3) with  $\sigma(\varepsilon) = \varepsilon^{1/2}$ , (A5) and (A6). For each  $\varepsilon \in (0, 1]$ , let  $(u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon)$  be the solution of  $(P)_\varepsilon$  obtained in Proposition 2.1. Then, there exists one triplet  $(u, \mu, \xi)$  such that*

$$\begin{aligned} u_\varepsilon &\rightarrow u \quad \text{strongly in } C([0, T]; V^*) \text{ and weakly star in } H^1(0, T; V^*) \cap L^\infty(0, T; H), \\ \varepsilon u_\varepsilon &\rightarrow 0 \quad \text{strongly in } L^\infty(0, T; V) \text{ and weakly in } L^2(0, T; W), \\ \pi_\varepsilon(u_\varepsilon) &\rightarrow 0 \quad \text{strongly in } L^\infty(0, T; H) \end{aligned}$$

and, at least for a subsequence,

$$\begin{aligned} \mu_\varepsilon &\rightarrow \mu \quad \text{weakly in } L^2(0, T; V), \\ \xi_\varepsilon &\rightarrow \xi \quad \text{weakly in } L^2(0, T; H) \end{aligned}$$

as  $\varepsilon \searrow 0$ . The triplet  $(u, \mu, \xi)$  is a weak solution of  $(P)$  and the component  $u$  is uniquely determined. Moreover, if  $\beta$  is single-valued, then  $\mu$  and  $\xi$  are also unique.

### 3 Uniform estimates

In this section, we obtain the uniform estimates useful to prove the convergence theorem. Throughout this section and the next Section 4, we will argue under the assumptions (A1)–(A5); the suitable modifications for the other set of assumptions of Theorem 2.3 will be discussed in Section 6.

#### 3.1 Approximate problem for $(P)_\varepsilon$

In order to obtain the uniform estimates with respect to  $\varepsilon \in (0, 1]$ , we need to consider a problem approximating  $(P)_\varepsilon$ , and this is actually a strategy to prove Proposition 2.1. Therefore, let us sketch the proof of Proposition 2.1 here. For each  $\lambda \in (0, 1]$ , consider the problem  $(P)_{\varepsilon, \lambda}$  which consists in finding the pair  $(u_{\varepsilon, \lambda}, \mu_{\varepsilon, \lambda})$  satisfying

$$\langle u'_{\varepsilon, \lambda}(t), z \rangle_{V^*, V} + \int_{\Omega} \nabla \mu_{\varepsilon, \lambda}(t) \cdot \nabla z dx = 0 \quad \text{for all } z \in V, \quad (3.1)$$

$$\mu_{\varepsilon, \lambda}(t) = \lambda u'_{\varepsilon, \lambda}(t) - \varepsilon \Delta u_{\varepsilon, \lambda}(t) + \beta_\lambda(u_{\varepsilon, \lambda}(t)) + \pi_\varepsilon(u_{\varepsilon, \lambda}(t)) - f(t) \quad \text{in } H, \quad (3.2)$$

for a.a.  $t \in (0, T)$ , and

$$u_{\varepsilon, \lambda}(0) = u_{0\varepsilon} \quad \text{in } H.$$

Here,  $\beta_\lambda$  is the Yosida approximation of  $\beta$  (see, e.g., [4, 8, 33]), that is,  $\beta_\lambda : \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$\beta_\lambda(r) := \frac{1}{\lambda}(r - J_\lambda(r)) := \frac{1}{\lambda}(r - (I + \lambda\beta)^{-1}(r)),$$

for all  $r \in \mathbb{R}$ , where the associated  $J_\lambda : \mathbb{R} \rightarrow \mathbb{R}$  is called the resolvent operator. It is well known that  $\beta_\lambda$  is the derivative of the Moreau–Yosida regularization  $\widehat{\beta}_\lambda : \mathbb{R} \rightarrow \mathbb{R}$  of  $\widehat{\beta}$ :

$$\widehat{\beta}_\lambda(r) := \inf_{s \in \mathbb{R}} \left\{ \frac{1}{2\lambda} |r - s|^2 + \widehat{\beta}(s) \right\} = \frac{1}{2\lambda} |r - J_\lambda(r)|^2 + \widehat{\beta}(J_\lambda(r)), \quad r \in \mathbb{R}.$$

The inequalities  $0 \leq \widehat{\beta}_\lambda(r) \leq \widehat{\beta}(r)$  hold for all  $r \in \mathbb{R}$ .

Based on available results (cf., e.g., [9, 34, 36]), it turns out that the problem  $(P)_{\varepsilon,\lambda}$  has the solution  $(u_{\varepsilon,\lambda}, \mu_{\varepsilon,\lambda})$ , with  $u_{\varepsilon,\lambda} \in H^1(0, T; H) \cap C([0, T]; V) \cap L^2(0, T; W)$  and  $\mu_{\varepsilon,\lambda} \in L^2(0, T; V)$ . Indeed, (3.1)–(3.2) is equivalent to an evolution equation in terms of the variable  $v_{\varepsilon,\lambda} := u_{\varepsilon,\lambda} - m_0$  in the subspace  $H_0 := \{z \in H : m(z) = 0\}$  of  $H$ . Then, we can adapt the results of [18] for doubly nonlinear evolution equations to show the existence.

Now, recalling the definition of  $\mathcal{N}$  (see (2.1)–(2.2)), we take  $z = 1$  in (3.1) and obtain  $u'_{\varepsilon,\lambda}(t) \in D(\mathcal{N})$ ; moreover, (3.1) can be rewritten as

$$\mathcal{N}u'_{\varepsilon,\lambda}(t) = m(\mu_{\varepsilon,\lambda}(t)) - \mu_{\varepsilon,\lambda}(t) \quad \text{in } V, \quad (3.3)$$

whence (3.2) is equivalent to

$$\mathcal{N}u'_{\varepsilon,\lambda}(t) - m(\mu_{\varepsilon,\lambda}(t)) + \lambda u'_{\varepsilon,\lambda}(t) - \varepsilon \Delta u_{\varepsilon,\lambda}(t) + \beta_\lambda(u_{\varepsilon,\lambda}(t)) + \pi_\varepsilon(u_{\varepsilon,\lambda}(t)) = f(t) \quad \text{in } H, \quad (3.4)$$

for a.a.  $t \in (0, T)$ . As  $(d/dt) \int_\Omega u_{\varepsilon,\lambda}(t) dx = 0$ , it turns out that

$$\frac{1}{|\Omega|} \int_\Omega u_{\varepsilon,\lambda}(t) dx = \frac{1}{|\Omega|} \int_\Omega u_{0\varepsilon} dx = m_0 \quad (3.5)$$

for a.a.  $t \in (0, T)$ , because  $m(u_{0\varepsilon}) = m_0$ .

## 3.2 Deduction of the estimates

The proof of the convergence theorem is based on the estimates, independent of  $\varepsilon$ , for the solutions of  $(P)_\varepsilon$ . Here, we derive the useful uniform estimates on the approximating problem  $(P)_{\varepsilon,\lambda}$  and, after stating and proving the series of next lemmas, we comment about the limit as  $\lambda \searrow 0$ .

**Lemma 3.1.** *There exists a positive constant  $M_1$  and two values  $\bar{\lambda}, \bar{\varepsilon} \in (0, 1]$ , depending only on the data, such that*

$$\begin{aligned} \int_0^t |u'_{\varepsilon,\lambda}(s)|_{V^*}^2 ds + 2\lambda \int_0^t |u'_{\varepsilon,\lambda}(s)|_H^2 ds + \varepsilon |\nabla u_{\varepsilon,\lambda}(t)|_{H^d}^2 \\ + |\widehat{\beta}_\lambda(u_{\varepsilon,\lambda}(t))|_{L^1(\Omega)} + \frac{c_1}{4} |u_{\varepsilon,\lambda}(t)|_H^2 \leq M_1 \end{aligned}$$

for all  $t \in [0, T]$ ,  $\lambda \in (0, \bar{\lambda}]$  and  $\varepsilon \in (0, \bar{\varepsilon}]$ .

**Proof.** In order to obtain the boundedness in  $L^\infty(0, T; H)$  for  $u_{\varepsilon,\lambda}$ , here we exploit the assumption (A2). About the other set of assumptions of Theorem 2.3, we will discuss the variation of this lemma in Section 6. We test (3.4) at time  $s \in (0, T)$  by  $u'_{\varepsilon,\lambda}(s) \in H$  and integrate the resultant over  $(0, t)$ . In view of (2.2), (2.3), and  $m(u'_{\varepsilon,\lambda}(s)) = 0$ , we deduce that

$$\begin{aligned} (\mathcal{N}u'_{\varepsilon,\lambda}(s), u'_{\varepsilon,\lambda}(s))_H &= \langle u'_{\varepsilon,\lambda}(s), \mathcal{N}u'_{\varepsilon,\lambda}(s) \rangle_{V^*, V} \\ &= \int_\Omega |\nabla \mathcal{N}u'_{\varepsilon,\lambda}(s)|^2 dx = |u'_{\varepsilon,\lambda}(s)|_{V^*}^2 \end{aligned}$$

and  $(m(\mu_{\varepsilon,\lambda}(s)), u'_{\varepsilon,\lambda}(s))_H = 0$  for a.a.  $s \in (0, T)$ . Therefore, we have

$$\begin{aligned} & \int_0^t |u'_{\varepsilon,\lambda}(s)|_{V^*}^2 ds + \lambda \int_0^t |u'_{\varepsilon,\lambda}(s)|_H^2 ds + \frac{\varepsilon}{2} |\nabla u_{\varepsilon,\lambda}(t)|_{H^d}^2 + \int_{\Omega} \widehat{\beta}_{\lambda}(u_{\varepsilon,\lambda}(t)) dx \\ & \leq \frac{\varepsilon}{2} |\nabla u_{0\varepsilon}|_{H^d}^2 + \int_{\Omega} \widehat{\beta}_{\lambda}(u_{0\varepsilon}) dx + \int_{\Omega} \widehat{\pi}_{\varepsilon}(u_{0\varepsilon}) dx - \int_{\Omega} \widehat{\pi}_{\varepsilon}(u_{\varepsilon,\lambda}(t)) dx \\ & \quad + \int_0^t \langle u'_{\varepsilon,\lambda}(s), f(s) \rangle_{V^*, V} ds \end{aligned} \quad (3.6)$$

for all  $t \in [0, T]$ , where  $\widehat{\pi}_{\varepsilon}$  is the primitive of  $\pi_{\varepsilon}$  defined by  $\widehat{\pi}_{\varepsilon}(r) := \int_0^r \pi_{\varepsilon}(\rho) d\rho$  for all  $r \in \mathbb{R}$ . Now, from (A2) it follows that

$$\widehat{\beta}_{\lambda}(r) = \frac{1}{2\lambda} |r - J_{\lambda}(r)|^2 + \widehat{\beta}(J_{\lambda}(r)) \geq \frac{1}{2\lambda} |r - J_{\lambda}(r)|^2 + c_1 |J_{\lambda}(r)|^2 - c_2$$

for all  $r \in \mathbb{R}$ . Therefore, taking  $\bar{\lambda} := \min\{1, 1/(2c_1)\}$  and using  $1/(4\bar{\lambda}) \geq c_1/2$ , we have

$$\begin{aligned} \int_{\Omega} \widehat{\beta}_{\lambda}(u_{\varepsilon,\lambda}(s)) dx &= \frac{1}{2} \int_{\Omega} \widehat{\beta}_{\lambda}(u_{\varepsilon,\lambda}(s)) dx + \frac{1}{4\bar{\lambda}} |u_{\varepsilon,\lambda}(s) - J_{\lambda}(u_{\varepsilon,\lambda}(s))|_H^2 \\ &\quad + \frac{c_1}{2} |J_{\lambda}(u_{\varepsilon,\lambda}(s))|_H^2 - \frac{c_2}{2} |\Omega| \\ &\geq \frac{1}{2} \int_{\Omega} \widehat{\beta}_{\lambda}(u_{\varepsilon,\lambda}(s)) dx + \frac{c_1}{4} |u_{\varepsilon,\lambda}(s)|_H^2 - \frac{c_2}{2} |\Omega| \end{aligned} \quad (3.7)$$

for a.a.  $s \in (0, T)$ . Next, recalling the Maclaurin expansion and (2.8) of (A3), we infer that

$$|\widehat{\pi}_{\varepsilon}(r)| \leq |\pi_{\varepsilon}(0)| |r| + \frac{1}{2} |\pi'_{\varepsilon}|_{L^{\infty}(\mathbb{R})} r^2 \leq c_3 \sigma(\varepsilon) (1 + r^2)$$

for all  $r \in \mathbb{R}$ . Now, from (A3) we deduce that there exists  $\bar{\varepsilon} \in (0, 1]$  such that  $\sigma(\varepsilon) \leq c_1/(8c_3(1 + |\Omega|))$  for all  $\varepsilon \in (0, \bar{\varepsilon}]$ . Thus, we have

$$- \int_{\Omega} \widehat{\pi}_{\varepsilon}(u_{\varepsilon,\lambda}(s)) dx \leq c_3 \sigma(\varepsilon) \int_{\Omega} (1 + |u_{\varepsilon,\lambda}(s)|^2) dx \leq \frac{c_1}{8} (1 + |u_{\varepsilon,\lambda}(s)|_H^2) \quad (3.8)$$

for a.a.  $s \in (0, T)$ . Moreover, using (2.9) of (A5) leads to

$$\begin{aligned} & \frac{\varepsilon}{2} |\nabla u_{0\varepsilon}|_{H^d}^2 + \int_{\Omega} \widehat{\beta}_{\lambda}(u_{0\varepsilon}) dx + \int_{\Omega} \widehat{\pi}_{\varepsilon}(u_{0\varepsilon}) dx \\ & \leq \frac{c_4}{2} + \int_{\Omega} \widehat{\beta}(u_{0\varepsilon}) dx + c_3 \sigma(\varepsilon) (1 + |\Omega|) (1 + |u_{0\varepsilon}(s)|_H^2) \leq \frac{3}{2} c_4 + \frac{c_1}{8} (1 + c_4). \end{aligned} \quad (3.9)$$

Thus, collecting (3.6)–(3.9), with the help of the Young inequality we arrive at

$$\begin{aligned} & \frac{1}{2} \int_0^t |u'_{\varepsilon,\lambda}(s)|_{V^*}^2 ds + \lambda \int_0^t |u'_{\varepsilon,\lambda}(s)|_H^2 ds + \frac{\varepsilon}{2} |\nabla u_{\varepsilon,\lambda}(t)|_{H^d}^2 \\ & \quad + \frac{1}{2} |\widehat{\beta}_{\lambda}(u_{\varepsilon,\lambda}(t))|_{L^1(\Omega)} + \frac{c_1}{8} |u_{\varepsilon,\lambda}(t)|_H^2 \\ & \leq \frac{c_2}{2} |\Omega| + \frac{3}{2} c_4 + \frac{c_1}{4} + \frac{c_1 c_4}{8} + \frac{1}{2} |f|_{L^2(0,T;V)}^2 \end{aligned} \quad (3.10)$$

for all  $t \in [0, T]$ . Thus, using (2.9) of (A5) we see that there exists a positive constant  $M_1$  depending only on  $c_1, c_2, c_4, |\Omega|$  and  $|f|_{L^2(0,T;V)}$ , independent of  $\varepsilon \in (0, \bar{\varepsilon}]$  and  $\lambda \in (0, \bar{\lambda}]$ , such that the aforementioned estimate holds.  $\square$

**Lemma 3.2.** *There exists a positive constant  $M_2$ , independent of  $\varepsilon \in (0, \bar{\varepsilon}]$  and  $\lambda \in (0, \bar{\lambda}]$ , such that*

$$\int_0^t |\beta_\lambda(u_{\varepsilon,\lambda}(s))|_{L^1(\Omega)}^2 ds \leq M_2$$

for all  $t \in [0, T]$ .

**Proof.** In view of (3.5), we have that  $u_{\varepsilon,\lambda}(s) - m_0 \in D(\mathcal{N})$  for a.a.  $s \in (0, T)$ . Let us recall the useful inequality proved in [26, Section 5, p. 908]: thanks to (A3) and (A5) (in particular, to the facts that  $m_0$  lies in the interior of  $D(\beta)$  and that  $\widehat{\beta}(0) = 0$ ), it turns out that there exist two positive constants  $c_5, c_6$  (depending on the position of  $m_0$ ) such that

$$\beta_\lambda(r)(r - m_0) \geq c_5 |\beta_\lambda(r)| - c_6 \quad \text{for all } r \in \mathbb{R}. \quad (3.11)$$

Now, we can test (3.4) at time  $s \in (0, T)$  by  $u_{\varepsilon,\lambda}(s) - m_0 \in D(\mathcal{N})$ . Then, we obtain

$$\begin{aligned} & \varepsilon |\nabla u_{\varepsilon,\lambda}(s)|_{H^d}^2 + (\beta_\lambda(u_{\varepsilon,\lambda}(s)), u_{\varepsilon,\lambda}(s) - m_0)_H \\ & \leq -(\mathcal{N}u'_{\varepsilon,\lambda}(s), u_{\varepsilon,\lambda}(s) - m_0)_H - \lambda(u'_{\varepsilon,\lambda}(s), u_{\varepsilon,\lambda}(s) - m_0)_H \\ & \quad - (\pi_\varepsilon(u_{\varepsilon,\lambda}(s)), u_{\varepsilon,\lambda}(s) - m_0)_H + (f(s), u_{\varepsilon,\lambda}(s) - m_0)_H \end{aligned} \quad (3.12)$$

for a.a.  $s \in (0, T)$ , because  $(m(u_{\varepsilon,\lambda}(s)), u_{\varepsilon,\lambda}(s) - m_0)_H = 0$ . Now, we know that there exists a positive constant  $c_7$  such that  $|z|_{V^*} \leq c_7 |z|_H$  for all  $z \in H$ , therefore

$$\begin{aligned} -(\mathcal{N}u'_{\varepsilon,\lambda}(s), u_{\varepsilon,\lambda}(s) - m_0)_H &= -\langle u_{\varepsilon,\lambda}(s) - m_0, \mathcal{N}u'_{\varepsilon,\lambda}(s) \rangle_{V^*, V} \\ &= (u'_{\varepsilon,\lambda}(s), u_{\varepsilon,\lambda}(s) - m_0)_{V^*} \\ &\leq c_7 |u'_{\varepsilon,\lambda}(s)|_{V^*} (|u_{\varepsilon,\lambda}(s)|_H + |m_0| |\Omega|) \end{aligned} \quad (3.13)$$

for a.a.  $s \in (0, T)$ . Next, we have that

$$-\lambda(u'_{\varepsilon,\lambda}(s), u_{\varepsilon,\lambda}(s) - m_0)_H \leq \lambda |u'_{\varepsilon,\lambda}(s)|_H (|u_{\varepsilon,\lambda}(s)|_H + |m_0| |\Omega|), \quad (3.14)$$

$$(f(s), u_{\varepsilon,\lambda}(s) - m_0)_H \leq |f(s)|_H (|u_{\varepsilon,\lambda}(s)|_H + |m_0| |\Omega|) \quad (3.15)$$

for a.a.  $s \in (0, T)$ . Additionally, it is straightforward to see that

$$\begin{aligned} & -(\pi_\varepsilon(u_{\varepsilon,\lambda}(s)), u_{\varepsilon,\lambda}(s) - m_0)_H \\ & \leq \int_\Omega (|\pi_\varepsilon(u_{\varepsilon,\lambda}(s)) - \pi_\varepsilon(0)| + |\pi_\varepsilon(0)|) |u_{\varepsilon,\lambda}(s) - m_0| dx \\ & \leq \int_\Omega (|\pi'_\varepsilon|_{L^\infty(\mathbb{R})} |u_{\varepsilon,\lambda}(s)| + |\pi_\varepsilon(0)|) |u_{\varepsilon,\lambda}(s) - m_0| dx \\ & \leq c_3 \int_\Omega (|u_{\varepsilon,\lambda}(s)| + 1) (|u_{\varepsilon,\lambda}(s)| + |m_0|) dx \\ & \leq c_8 (1 + |u_{\varepsilon,\lambda}(s)|_H^2) \end{aligned} \quad (3.16)$$

for a.a.  $s \in (0, T)$  and for some constant  $c_8 > 0$  depending only on  $c_3$ ,  $|\Omega|$  and  $|m_0|$ . Then, collecting (3.12)–(3.16) and using (3.11), we deduce that

$$\begin{aligned} |\beta_\lambda(u_{\varepsilon,\lambda}(s))|_{L^1(\Omega)} &\leq \frac{c_6}{c_5}|\Omega| + \frac{c_7}{c_5}|u'_{\varepsilon,\lambda}(s)|_{V^*} \left( |u_{\varepsilon,\lambda}(s)|_H + |m_0||\Omega| \right) \\ &\quad + \frac{1}{c_5} \left( \lambda |u'_{\varepsilon,\lambda}(s)|_H + |f(s)|_H \right) \left( |u_{\varepsilon,\lambda}(s)|_H + |m_0||\Omega| \right) \\ &\quad + \frac{c_8}{c_5} \left( 1 + |u_{\varepsilon,\lambda}(s)|_H^2 \right) \end{aligned} \quad (3.17)$$

for a.a.  $s \in (0, T)$ . Now, we square the sides of (3.17), integrate the resultant over  $(0, T)$ , and take advantage of Lemma 3.1. Thus, we easily find a positive constant  $M_2$ , depending only on  $M_1$ ,  $c_5$ ,  $c_6$ ,  $c_7$ ,  $c_8$ ,  $|\Omega|$ ,  $T$ ,  $|m_0|$  and  $|f|_{L^2(0,T;H)}$ , such that the assertion of the lemma follows.  $\square$

**Lemma 3.3.** *There exists a positive constants  $M_3$ , independent of  $\varepsilon \in (0, \bar{\varepsilon}]$  and  $\lambda \in (0, \bar{\lambda}]$ , such that*

$$\int_0^t |m(\mu_{\varepsilon,\lambda}(s))|^2 ds \leq M_3$$

for all  $t \in [0, T]$ .

**Proof.** Recalling the assumption (A3), we have that

$$\begin{aligned} |\pi_\varepsilon(u_{\varepsilon,\lambda}(s))|^2 &\leq \left\{ |\pi'_\varepsilon|_{L^\infty(\mathbb{R})} |u_{\varepsilon,\lambda}(s)| + |\pi_\varepsilon(0)| \right\}^2 \\ &\leq 2c_3^2 \sigma(\varepsilon)^2 \left( 1 + |u_{\varepsilon,\lambda}(s)|^2 \right) \quad \text{a.e. in } \Omega, \end{aligned} \quad (3.18)$$

for a.a.  $s \in (0, T)$ . Now, integrating (3.4) over  $\Omega$  we can also exploit the Neumann homogeneous boundary condition for  $u_{\varepsilon,\lambda}$  (hidden in the  $L^2(0, T; W)$  regularity). By squaring and using (3.18), we easily obtain

$$\begin{aligned} |m(\mu_{\varepsilon,\lambda}(s))|^2 &\leq \frac{3}{|\Omega|^2} \left\{ |\beta_\lambda(u_{\varepsilon,\lambda}(s))|_{L^1(\Omega)}^2 + |\Omega| |\pi_\varepsilon(u_{\varepsilon,\lambda}(s))|_H^2 + |\Omega| |f(s)|_H^2 \right\} \\ &\leq \frac{3}{|\Omega|^2} \left\{ |\beta_\lambda(u_{\varepsilon,\lambda}(s))|_{L^1(\Omega)}^2 + 2|\Omega| c_3^2 \left( |\Omega| + |u_{\varepsilon,\lambda}(s)|_H^2 \right) + |\Omega| |f(s)|_H^2 \right\} \end{aligned}$$

for a.a.  $s \in (0, T)$ , because  $\sigma(\varepsilon) \leq 1$ . Thus, by integrating over  $(0, t)$ , the existence of a positive constant  $M_3$  depending only on  $M_1$ ,  $M_2$ ,  $c_3$ ,  $|\Omega|$ ,  $T$  and  $|f|_{L^2(0,T;H)}$  follows.  $\square$

**Lemma 3.4.** *There exists a positive constant  $M_4$ , independent of  $\varepsilon \in (0, \bar{\varepsilon}]$  and  $\lambda \in (0, \bar{\lambda}]$ , such that*

$$\int_0^t |\mu_{\varepsilon,\lambda}(s)|_V^2 ds \leq M_4$$

for all  $t \in [0, T]$ .

**Proof.** By virtue of (2.3), (2.4), (3.3), and the fact  $m(u'_{\varepsilon,\lambda}(s)) = 0$ , we have that

$$\begin{aligned} \int_0^t |\mu_{\varepsilon,\lambda}(s)|_V^2 ds &\leq 2 \int_0^t |m(\mu_{\varepsilon,\lambda}(s))|_V^2 ds + 2 \int_0^t |\mathcal{N}u'_{\varepsilon,\lambda}(s)|_V^2 ds \\ &\leq 2 \int_0^t |m(\mu_{\varepsilon,\lambda}(s))|_H^2 ds + 2c_P \int_0^t |\nabla \mathcal{N}u'_{\varepsilon,\lambda}(s)|_{H^d}^2 ds \\ &\leq 2|\Omega| \int_0^t |m(\mu_{\varepsilon,\lambda}(s))|^2 ds + 2c_P \int_0^t |u'_{\varepsilon,\lambda}(s)|_{V^*}^2 ds \leq M_4 \end{aligned}$$

for all  $t \in [0, T]$ , where (see Lemmas 3.1 and 3.3)  $M_4$  is a positive constant depending only on  $M_1$ ,  $M_3$ ,  $c_P$  and  $|\Omega|$ .  $\square$

**Lemma 3.5.** *There exist two positive constants  $M_5$  and  $M_6$ , independent of  $\varepsilon \in (0, \bar{\varepsilon}]$  and  $\lambda \in (0, \bar{\lambda}]$ , such that*

$$\int_0^t |\beta_\lambda(u_{\varepsilon,\lambda}(s))|_H^2 ds \leq M_5, \quad \int_0^t |\varepsilon u_{\varepsilon,\lambda}(s)|_W^2 ds \leq M_6$$

for all  $t \in [0, T]$ .

**Proof.** We test (3.2) at time  $s \in (0, T)$  by  $\beta_\lambda(u_{\varepsilon,\lambda}(s)) \in V$ . As

$$-\varepsilon(\Delta u_{\varepsilon,\lambda}(s), \beta_\lambda(u_{\varepsilon,\lambda}(s)))_H = \varepsilon \int_\Omega \beta'_\lambda(u_{\varepsilon,\lambda}(s)) |\nabla u_{\varepsilon,\lambda}(s)|^2 dx \geq 0,$$

due to the monotonicity of  $\beta_\lambda$ , we obtain

$$\begin{aligned} &|\beta_\lambda(u_{\varepsilon,\lambda}(s))|_H^2 \\ &\leq (\mu_{\varepsilon,\lambda}(s) - \lambda u'_{\varepsilon,\lambda}(s) - \pi_\varepsilon(u_{\varepsilon,\lambda}(s)) + f(s), \beta_\lambda(u_{\varepsilon,\lambda}(s)))_H \\ &\leq \frac{1}{2} |\beta_\lambda(u_{\varepsilon,\lambda}(s))|_H^2 + 2 \left( |\mu_{\varepsilon,\lambda}(s)|_H^2 + \lambda^2 |u'_{\varepsilon,\lambda}(s)|_H^2 + |\pi_\varepsilon(u_{\varepsilon,\lambda}(s))|_H^2 + |f(s)|_H^2 \right) \end{aligned}$$

for a.a.  $s \in (0, T)$ . Now, integrating over  $(0, t)$  with respect to  $s$ , from (3.18) and Lemmas 3.1 and 3.4 it follows that

$$\begin{aligned} \int_0^t |\beta_\lambda(u_{\varepsilon,\lambda}(s))|_H^2 ds &\leq 4 \int_0^t |\mu_{\varepsilon,\lambda}(s)|_H^2 ds + 4\lambda^2 \int_0^t |u'_{\varepsilon,\lambda}(s)|_H^2 ds \\ &\quad + 8c_3^2 \left( |\Omega|T + \int_0^t |u_{\varepsilon,\lambda}(s)|_H^2 ds \right) + 4 \int_0^t |f(s)|_H^2 ds \leq M_5 \end{aligned}$$

for all  $t \in [0, T]$ , where  $M_5$  is a positive constant depending only on  $M_1$ ,  $M_4$ ,  $c_1$ ,  $c_3$ ,  $|\Omega|$ ,  $T$  and  $|f|_{L^2(0,T;H)}$ . At last, by comparison in (3.2) we deduce that

$$\begin{aligned} \int_0^t |\varepsilon \Delta u_{\varepsilon,\lambda}(s)|_H^2 ds &\leq 5 \int_0^t |\mu_{\varepsilon,\lambda}(s)|_H^2 ds + 5\lambda^2 \int_0^t |u'_{\varepsilon,\lambda}(s)|_H^2 ds + 5 \int_0^t |\beta_\lambda(u_{\varepsilon,\lambda}(s))|_H^2 ds \\ &\quad + 10c_3^2 \left( |\Omega|T + \int_0^t |u_{\varepsilon,\lambda}(s)|_H^2 ds \right) + 5 \int_0^t |f(s)|_H^2 ds \end{aligned}$$

whence, by the boundedness properties stated in Lemma 3.1 and standard elliptic regularity results, we infer

$$\int_0^t |\varepsilon u_{\varepsilon, \lambda}(s)|_W^2 ds \leq M_6$$

for all  $t \in [0, T]$ , where  $M_6$  is a positive constant having the same dependencies as  $M_5$ .  $\square$

Here we are: by using the uniform estimates stated in Lemmas from 3.1 to 3.5, we can pass to the limit in the approximate problem  $(P)_{\varepsilon, \lambda}$  as  $\lambda \searrow 0$ , and, with the help of monotonicity arguments, recover a solution  $(u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon)$  to  $(P)_\varepsilon$ . For a fixed  $\varepsilon \in (0, \bar{\varepsilon}]$ , the solution component  $u_\varepsilon$  is uniquely determined and, if  $\beta$  is single-valued,  $\mu_\varepsilon$  and  $\xi_\varepsilon$  are unique as well (let us quote, e.g., [9, 18, 34, 36] for the involved results). Moreover, thanks to the previous lemmas, we see that, on the procedure of the limit as  $\lambda \searrow 0$ , all the estimates (2.17)–(2.18) hold for  $(u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon)$ . These estimates turn out to be the key ingredient for the convergence result.

## 4 Proof of the convergence theorem

In this section, we prove the convergence theorem under the assumptions (A1)–(A5).

**Proof of Theorem 2.3 (first part).** Using the estimates (2.17)–(2.18), we see that there exist a subsequence  $\{\varepsilon_k\}_{k \in \mathbb{N}}$ , with  $\varepsilon_k \searrow 0$  as  $k \nearrow \infty$ , and some limit functions  $u \in H^1(0, T; V^*) \cap L^\infty(0, T; H)$ ,  $\mu \in L^2(0, T; V)$  and  $\xi \in L^2(0, T; H)$  such that

$$u_{\varepsilon_k} \rightarrow u \quad \text{weakly star in } H^1(0, T; V^*) \cap L^\infty(0, T; H), \quad (4.1)$$

$$\varepsilon_k u_{\varepsilon_k} \rightarrow 0 \quad \text{strongly in } L^\infty(0, T; V), \quad (4.2)$$

$$\mu_{\varepsilon_k} \rightarrow \mu \quad \text{weakly in } L^2(0, T; V), \quad (4.3)$$

$$\xi_{\varepsilon_k} \rightarrow \xi \quad \text{weakly in } L^2(0, T; H) \quad (4.4)$$

as  $k \nearrow \infty$ . From (4.1) and the well-known Ascoli–Arzela theorem (see, e.g., [42, Section 8, Corollary 4]), we deduce that

$$u_{\varepsilon_k} \rightarrow u \quad \text{strongly in } C([0, T]; V^*); \quad (4.5)$$

moreover, (4.2) and the boundedness property in (2.18) imply that

$$\varepsilon_k u_{\varepsilon_k} \rightarrow 0 \quad \text{weakly in } L^2(0, T; W)$$

as  $k \nearrow \infty$ . With the help of the assumption (A3) (see also (3.18)), we have that

$$|\pi_{\varepsilon_k}(u_{\varepsilon_k})| \leq 2c_3 \sigma(\varepsilon_k)(1 + |u_{\varepsilon_k}|) \quad \text{a.e. in } Q,$$

and consequently (4.1) enables us to infer that

$$\pi_\varepsilon(u_{\varepsilon_k}) \rightarrow 0 \quad \text{strongly in } L^\infty(0, T; H) \quad (4.6)$$



as  $k \nearrow \infty$ . Now, using (4.1)–(4.4) and (4.6), we can pass to the limit in (2.13) and (2.14) obtaining (2.19) and the equality in (2.20) for the limit functions  $u$ ,  $\mu$  and  $\xi$ . Note that the function  $u$  is weakly continuous from  $[0, T]$  to  $H$ , since

$$u \in C([0, T]; V^*) \cap L^\infty(0, T; H).$$

Then, the initial condition (2.21) makes sense and follows from (2.16) and (A5). Moreover, the solution component  $\xi$  belongs to  $L^2(0, T; V)$ , due to  $\xi = \mu + f$  and (A4), even though (4.4) holds true just in  $L^2(0, T; H)$ .

It remains to check that  $\xi \in \beta(u)$  a.e. in  $Q$ . To this aim, it suffices to recall that

$$u_{\varepsilon_k} \rightarrow u, \quad \xi_{\varepsilon_k} \rightarrow \xi \quad \text{weakly in } L^2(0, T; H)$$

as  $k \nearrow \infty$  and verify that

$$\limsup_{k \nearrow \infty} \int_0^T (\xi_{\varepsilon_k}(t), u_{\varepsilon_k}(t))_H dt \leq \int_0^T (\xi(t), u(t))_H dt, \quad (4.7)$$

(cf. [4, Proposition 2.2, p. 38]). In order to show this, we test (2.20) by  $u_{\varepsilon_k}(t)$  and integrate over  $(0, T)$ . Note that

$$\begin{aligned} & \int_0^T (\xi_{\varepsilon_k}(t), u_{\varepsilon_k}(t))_H dt \\ &= \int_0^T (\mu_{\varepsilon_k}(t) + f(t), u_{\varepsilon_k}(t))_H dt - \varepsilon_k \int_0^T |\nabla u_{\varepsilon_k}(t)|_{H^d}^2 dt - \int_0^T (\pi_\varepsilon(u_{\varepsilon_k}(t)), u_{\varepsilon_k}(t))_H dt \\ &\leq \int_0^T \langle u_{\varepsilon_k}(t), \mu_{\varepsilon_k}(t) + f(t) \rangle_{V^*, V} dt - \int_0^T (\pi_\varepsilon(u_{\varepsilon_k}(t)), u_{\varepsilon_k}(t))_H dt \end{aligned}$$

and two terms on the last line converge. Namely, on account of (4.3) and (4.5), we have

$$\begin{aligned} \lim_{k \nearrow \infty} \int_0^T \langle u_{\varepsilon_k}(t), \mu_{\varepsilon_k}(t) + f(t) \rangle_{V^*, V} dt &= \int_0^T \langle u(t), \mu(t) + f(t) \rangle_{V^*, V} dt \\ &= \int_0^T (u(t), \mu(t) + f(t))_H dt = \int_0^T (\xi(t), u(t))_H dt, \end{aligned}$$

as  $\mu + f = \xi$  a.e. in  $Q$ . Moreover, (4.1) and (4.6) imply that

$$\lim_{k \nearrow \infty} \int_0^T (\pi_\varepsilon(u_{\varepsilon_k}(t)), u_{\varepsilon_k}(t))_H dt = 0.$$

Thus, we have checked (4.7) and completely proved that the limit triplet  $(u, \mu, \xi)$  yields one solution of (P).

We now verify that the solution component  $u$  of the problem (P) is unique. This part follows closely [32, Proposition 2.1]. Assume by contradiction that there are two solutions  $(u_i, \mu_i, \xi_i)$ ,  $i = 1, 2$ , and take the difference of the respective equations (2.19). Then, we can take  $z = \mathcal{N}(u_1(s) - u_2(s))$  since  $m(u_1(s) - u_2(s)) = 0$ : indeed, for  $i = 1, 2$  we have

$\langle u_i(s), 1 \rangle_{V^*, V} = \langle u_0, 1 \rangle_{V^*, V}$  for all  $s \in [0, T]$ , directly from (2.19) and (2.21). Hence, as  $\mu_1 - \mu_2 = \xi_1 - f - (\xi_2 - f)$  a.e. in  $Q$  we obtain

$$\frac{1}{2} |u_1(t) - u_2(t)|_{V^*}^2 + \int_0^t (\xi_1(s) - \xi_2(s), u_1(s) - u_2(s))_H ds = 0$$

for all  $t \in [0, T]$ . Then, from (2.20) and the monotonicity of  $\beta$  the second term is nonnegative, so that

$$|u_1(t) - u_2(t)|_{V^*}^2 = 0$$

for all  $t \in [0, T]$  and the uniqueness property follows. In general,  $\xi$  and  $\mu$  are not uniquely determined; indeed, if  $\beta$  is multivalued (as in the case of Example 3) it may happen that we could add to both  $\xi$  and  $\mu$  a function depending only on time and preserve the validity of (2.19) and (2.20). However, if the graph  $\beta$  is single-valued in its domain (like in Examples 1, 2, 4), then  $\xi$  is completely determined from the inclusion in (2.20), and so is  $\mu$ . Anyway, being  $u$  unique, it turns out that the convergences in (4.1)–(4.2) and (4.5)–(4.6) hold for all the families as  $\varepsilon \searrow 0$  and not only for a subsequence. Then, the assertion of Theorem 2.3 is completely proved.  $\square$

**Remark 4.1.** The assertion of Theorem 2.3 is a consequence of our uniform estimates proved for all  $\varepsilon \in (0, \bar{\varepsilon}]$ , as the reader easily realizes from the proof. The same reader can wonder whether the problem  $(P)_\varepsilon$  has a solution for  $\varepsilon \in (\bar{\varepsilon}, 1]$  (cf. our statement of Proposition 2.1). The answer is yes, although the basic estimates leading to the existence of the solutions are different from the ones detailed here. In order to see which kind of approach could be followed, we refer the reader to, e.g., [10] where a more general problem is fully investigated.

## 5 Error estimate

We are now going to state the error estimate, still under the assumptions (A1)–(A5) but with some reinforcement. Indeed, we improve the requirement (2.8) in (A3) by prescribing that

$$|\pi_\varepsilon(0)| + |\pi'_\varepsilon|_{L^\infty(\mathbb{R})} \leq c_3 \varepsilon^{1/2} \quad \text{for all } \varepsilon \in (0, 1], \quad (5.1)$$

namely  $\sigma(\varepsilon) := \varepsilon^{1/2}$ . Moreover, in the framework of (A5) we additionally assume that

$$|u_{0\varepsilon} - u_0|_{V^*} \leq c_9 \varepsilon^{1/4} \quad \text{for all } \varepsilon \in (0, 1], \quad (5.2)$$

for some positive constant  $c_9$ . We observe that if we take  $u_{0\varepsilon}$  exactly as in the choice postulated in the Appendix, then we can find a constant  $C > 0$  such that

$$|u_{0\varepsilon} - u_0|_{V^*}^2 \leq \varepsilon \langle u_0 - u_{0\varepsilon}, u_{0\varepsilon} \rangle_{V^*, V} \leq C\varepsilon \quad \text{for all } \varepsilon \in (0, 1].$$

Thus, we have the bound (5.2) even with  $\varepsilon^{1/2}$ .

**Theorem 5.1.** *Assume (A1)–(A5) with (5.1) and (5.2). For  $\varepsilon \in (0, \bar{\varepsilon}]$ , let  $(u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon)$  be a solution of problem  $(P)_\varepsilon$  and let  $(u, \mu, \xi)$  solve problem  $(P)$ . Then, there exists a constant  $C^* > 0$ , depending only on the data, such that*

$$|u_\varepsilon - u|_{C([0,T];V^*)}^2 + \int_0^T (\xi_\varepsilon(s) - \xi(s), u_\varepsilon(s) - u(s))_H ds \leq C^* \varepsilon^{1/3} \quad (5.3)$$

for all  $\varepsilon \in (0, \bar{\varepsilon}]$ .

Observe that the elements  $u_\varepsilon$  and  $u$  appearing in the error estimate (5.3) are uniquely determined.

**Proof of Theorem 5.1.** For  $\varepsilon \in (0, \bar{\varepsilon}]$ , let us use (3.2) at the level of the  $\lambda$ -approximation. We rewrite (3.2) as

$$-\varepsilon \Delta u_{\varepsilon,\lambda} + \beta_\lambda(u_{\varepsilon,\lambda}) = -\lambda u'_{\varepsilon,\lambda} + \mu_{\varepsilon,\lambda} + f - \pi_\varepsilon(u_{\varepsilon,\lambda}) \quad \text{a.e. in } Q,$$

and we test it by  $-\varepsilon^\alpha \Delta u_{\varepsilon,\lambda}$  with  $\alpha > 0$ . Then, since

$$\int_0^T (\beta_\lambda(u_{\varepsilon,\lambda}(s)), -\varepsilon^\alpha \Delta u_{\varepsilon,\lambda}(s))_H ds = \varepsilon^\alpha \int_0^T \int_\Omega \beta'_\lambda(u_{\varepsilon,\lambda}(s)) |\nabla u_{\varepsilon,\lambda}(s)|^2 dx ds \geq 0.$$

and

$$\begin{aligned} \int_0^T (-\lambda u'_{\varepsilon,\lambda}(s), -\varepsilon^\alpha \Delta u_{\varepsilon,\lambda}(s))_H ds &= -\lambda \varepsilon^\alpha \int_0^T \frac{1}{2} \frac{d}{ds} \int_\Omega |\nabla u_{\varepsilon,\lambda}(s)|^2 dx ds \\ &= -\lambda \varepsilon^\alpha \left( \frac{1}{2} \int_\Omega |\nabla u_{\varepsilon,\lambda}(T)|^2 dx - \frac{1}{2} \int_\Omega |\nabla u_{0\varepsilon}|^2 dx \right) \\ &\leq \frac{\lambda \varepsilon^\alpha}{2} \int_\Omega |\nabla u_{0\varepsilon}|^2 dx = \frac{\lambda \varepsilon^\alpha}{2} |\nabla u_{0\varepsilon}|_{H^d}^2, \end{aligned}$$

using the Young inequality, we obtain

$$\begin{aligned} &\varepsilon^{1+\alpha} \int_0^T |\Delta u_{\varepsilon,\lambda}(s)|_H^2 ds \\ &\leq \int_0^T (\mu_{\varepsilon,\lambda}(s) + f(s) - \pi_\varepsilon(u_{\varepsilon,\lambda}(s)), -\varepsilon^\alpha \Delta u_{\varepsilon,\lambda}(s))_H ds + \frac{\lambda \varepsilon^\alpha}{2} |\nabla u_{0\varepsilon}|_{H^d}^2 \\ &= \int_0^T (\nabla(\mu_{\varepsilon,\lambda}(s) + f(s)), \varepsilon^\alpha \nabla u_{\varepsilon,\lambda}(s))_H ds \\ &\quad - \int_0^T (\nabla \pi_\varepsilon(u_{\varepsilon,\lambda}(s)), \varepsilon^\alpha \nabla u_{\varepsilon,\lambda}(s))_H ds + \frac{\lambda \varepsilon^\alpha}{2} |\nabla u_{0\varepsilon}|_{H^d}^2 \\ &\leq \frac{1}{2} \int_0^T |\nabla(\mu_{\varepsilon,\lambda}(s) + f(s))|_{H^d}^2 ds + \frac{1}{2} \int_0^T |\pi'_\varepsilon(u_{\varepsilon,\lambda}(s)) \nabla u_{\varepsilon,\lambda}(s)|_{H^d}^2 ds \\ &\quad + \varepsilon^{2\alpha} \int_0^T |\nabla u_{\varepsilon,\lambda}(s)|_{H^d}^2 ds + \frac{\lambda \varepsilon^\alpha}{2} |\nabla u_{0\varepsilon}|_{H^d}^2. \end{aligned}$$

Then, in view of (5.1) we find out that

$$\begin{aligned} \varepsilon^{1+\alpha} \int_0^T |\Delta u_{\varepsilon,\lambda}(s)|_H^2 ds &\leq |\mu_{\varepsilon,\lambda}|_{L^2(0,T;V)}^2 + |f|_{L^2(0,T;V)}^2 + \frac{1}{2} c_3^2 T \varepsilon |\nabla u_{\varepsilon,\lambda}|_{L^\infty(0,T;H^d)}^2 \\ &\quad + \varepsilon^{2\alpha} \int_0^T |\nabla u_{\varepsilon,\lambda}(s)|_{H^d}^2 ds + \frac{\lambda \varepsilon^\alpha}{2} |\nabla u_{0\varepsilon}|_{H^d}^2. \end{aligned} \quad (5.4)$$

Now, we know that  $u_{\varepsilon,\lambda} \in L^2(0,T;W)$ , therefore

$$\begin{aligned} \varepsilon^{2\alpha} \int_0^T |\nabla u_{\varepsilon,\lambda}(s)|_{H^d}^2 ds &= \varepsilon^{2\alpha} \int_0^T (u_{\varepsilon,\lambda}(s), -\Delta u_{\varepsilon,\lambda}(s))_H ds \\ &\leq \frac{1}{2} \int_0^T |u_{\varepsilon,\lambda}(s)|_H^2 ds + \frac{\varepsilon^{4\alpha}}{2} \int_0^T |\Delta u_{\varepsilon,\lambda}(s)|_H^2 ds. \end{aligned} \quad (5.5)$$

If, we devise  $\varepsilon^{4\alpha} \leq \varepsilon^{1+\alpha}$ , that is  $\alpha = 1/3$ , then we recall Lemmas 3.1 and 3.4 and point out that (5.4) and (5.5) imply

$$\begin{aligned} \varepsilon^{4/3} \int_0^T |\Delta u_{\varepsilon,\lambda}(s)|_H^2 ds &\leq \left( 2M_4 + 2|f|_{L^2(0,T;V)}^2 + c_3^2 T M_1 + \frac{4TM_1}{c_1} \right) + \lambda \varepsilon^\alpha |\nabla u_{0\varepsilon}|_{H^d}^2 \\ &=: C_1^* + \lambda \varepsilon^\alpha |\nabla u_{0\varepsilon}|_{H^d}^2. \end{aligned} \quad (5.6)$$

The estimate (5.6) works for the approximate solution  $u_{\varepsilon,\lambda}$ . However, thanks to Proposition 2.1 and the proof of the convergence theorem, which was treated in the previous section,  $(u_{\varepsilon,\lambda}, \mu_{\varepsilon,\lambda}, \beta_\lambda(u_{\varepsilon,\lambda}))$  converges to  $(u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon)$  as  $\lambda \searrow 0$  and, passing to the limit in (5.6), we recover the key estimate

$$\varepsilon^{4/3} \int_0^T |\Delta u_\varepsilon(s)|_H^2 ds \leq C_1^*. \quad (5.7)$$

Next, take the difference between (2.13) and (2.19), then we have

$$\langle u'_\varepsilon(s) - u'(s), z \rangle_{V^*,V} + \int_\Omega \nabla(\mu_\varepsilon(s) - \mu(s)) \cdot \nabla z dx = 0 \quad \text{for all } z \in V,$$

for a.a.  $s \in (0, T)$ . We can choose  $z = \mathcal{N}(u_\varepsilon(s) - u(s))$  because (cf. (A5))  $m(u_\varepsilon(s) - u(s)) = m(u_{0\varepsilon}) - m(u_0) = 0$ ; then we obtain

$$\frac{1}{2} \frac{d}{ds} |u_\varepsilon(s) - u(s)|_{V^*}^2 + \langle u_\varepsilon(s) - u(s), \mu_\varepsilon(s) - \mu(s) \rangle_{V^*,V} = 0$$

for a.a.  $s \in (0, T)$ . Now, by integrating over  $(0, t)$ , owing to (2.14) and (2.20) we infer the following equality:

$$\begin{aligned} &\frac{1}{2} |u_\varepsilon(t) - u(t)|_{V^*}^2 + \int_0^t (u_\varepsilon(s) - u(s), \xi_\varepsilon(s) - \xi(s))_H ds \\ &= \frac{1}{2} |u_{0\varepsilon} - u_0|_{V^*}^2 + \varepsilon \int_0^t (\Delta u_\varepsilon(s), u_\varepsilon(s) - u(s))_H ds \\ &\quad - \int_0^t (\pi_\varepsilon(u_\varepsilon(s)), u_\varepsilon(s) - u(s))_H ds, \end{aligned} \quad (5.8)$$

holding for all  $t \in [0, T]$ . Note that, by using Lemma 3.1 and (5.7), one can find a positive constant  $C_2^*$ , depending only on  $c_1$ ,  $M_1$ ,  $C_1^*$  and  $T$ , such that

$$\begin{aligned} & \varepsilon \int_0^T (\Delta u_\varepsilon(s), u_\varepsilon(s) - u(s))_H ds \\ & \leq \varepsilon^{1/3} \left\{ \varepsilon^{4/3} \int_0^T |\Delta u_\varepsilon(s)|_H^2 ds \right\}^{1/2} \left\{ \int_0^T |u_\varepsilon(s) - u(s)|_H^2 ds \right\}^{1/2} \leq C_2^* \varepsilon^{1/3}. \end{aligned} \quad (5.9)$$

Next, on account of Lemma 3.1 and (5.1), there is a positive constant  $C_3^*$ , depending only on  $c_1$ ,  $c_3$ ,  $M_1$ ,  $|\Omega|$  and  $T$ , such that

$$\begin{aligned} & - \int_0^T (\pi_\varepsilon(u_\varepsilon(s)), u_\varepsilon(s) - u(s))_H ds \\ & \leq |\pi'_\varepsilon|_{L^\infty(\mathbb{R})} \left\{ \int_0^T |u_\varepsilon(s)|_H^2 ds \right\}^{1/2} \left\{ \int_0^T |u_\varepsilon(s) - u(s)|_H^2 ds \right\}^{1/2} \\ & \quad + |\pi_\varepsilon(0)| (|\Omega|T)^{1/2} \left\{ \int_0^T |u_\varepsilon(s) - u(s)|_H^2 ds \right\}^{1/2} \leq C_3^* \varepsilon^{1/2}. \end{aligned} \quad (5.10)$$

Thus, collecting (5.8)–(5.10) and applying (5.2) lead to

$$|u_\varepsilon - u|_{C([0,T];V^*)}^2 \leq c_9^2 \varepsilon^{1/2} + 2C_2^* \varepsilon^{1/3} + 2C_3^* \varepsilon^{1/2},$$

and

$$\int_0^T (u_\varepsilon(s) - u(s), \xi_\varepsilon(s) - \xi(s))_H ds \leq \frac{1}{2} c_9^2 \varepsilon^{1/2} + C_2^* \varepsilon^{1/3} + C_3^* \varepsilon^{1/2},$$

that is, there exists  $C^* > 0$  such that the error estimate (5.3) holds.  $\square$

**Remark 5.2.** If  $\beta$  is Lipschitz continuous, then (5.3) gives us the additional information

$$\int_0^T |\xi_\varepsilon(s) - \xi(s)|_H^2 ds \leq C_\beta \int_0^T (\xi_\varepsilon(s) - \xi(s), u_\varepsilon(s) - u(s))_H ds \leq C_\beta C^* \varepsilon^{1/3},$$

due to the monotonicity of  $\beta$ . Here,  $C_\beta$  denotes a Lipschitz constant for  $\beta$ .

## 6 Improvement of the results

Throughout this section, we assume that (A1), (A3) with  $\sigma(\varepsilon) = \varepsilon^{1/2}$  (that is, (5.1)), (A5) and (A6) hold. Let us note that the assumption (A6) implies that

$$\begin{cases} -\Delta f(t) = g(t) & \text{a.e. in } \Omega, \\ \partial_\nu f(t) = 0 & \text{a.e. in } \Gamma, \end{cases}$$

for a.a.  $t \in (0, T)$  (cf. (2.6)). We point out that (A2) is no longer in use and (A4) is covered by (A6).

**Proof of Theorem 2.3 (final part).** Referring to the first part of the proof given in Section 3, here we have to modify the proof of Lemma 3.1. We take  $z := u_{\varepsilon,\lambda}(s)$  in (3.1) obtaining

$$\langle u'_{\varepsilon,\lambda}(s), u_{\varepsilon,\lambda}(s) \rangle_{V^*,V} + \int_{\Omega} \nabla \mu_{\varepsilon,\lambda}(s) \cdot \nabla u_{\varepsilon,\lambda}(s) dx = 0 \quad (6.1)$$

for a.a.  $s \in (0, T)$ ; on the other hand, we test (3.2) by  $-\Delta u_{\varepsilon,\lambda}(s)$  and exploit (A6) to deduce that

$$\begin{aligned} & \int_{\Omega} \nabla \mu_{\varepsilon,\lambda}(s) \cdot \nabla u_{\varepsilon,\lambda}(s) dx \\ &= \frac{\lambda}{2} \frac{d}{ds} |\nabla u_{\varepsilon,\lambda}(s)|_{H^d}^2 + \varepsilon |\Delta u_{\varepsilon,\lambda}(s)|_H^2 + \int_{\Omega} \beta'_{\lambda}(u_{\varepsilon,\lambda}(s)) |\nabla u_{\varepsilon,\lambda}(s)|^2 dx \\ & \quad - (\pi_{\varepsilon}(u_{\varepsilon,\lambda}(s)), \Delta u_{\varepsilon,\lambda}(s))_H + (\Delta f(s), u_{\varepsilon,\lambda}(s))_H \end{aligned} \quad (6.2)$$

for a.a.  $s \in (0, T)$ . By virtue of (A3) and the Young inequality, we have that

$$\begin{aligned} (\pi_{\varepsilon}(u_{\varepsilon,\lambda}(s)), \Delta u_{\varepsilon,\lambda}(s))_H &\leq \int_{\Omega} (|\pi'_{\varepsilon}|_{L^{\infty}(\mathbb{R})} |u_{\varepsilon,\lambda}(s)| + |\pi_{\varepsilon}(0)|) |\Delta u_{\varepsilon,\lambda}(s)| dx \\ &\leq \int_{\Omega} c_3 \varepsilon^{1/2} (|u_{\varepsilon,\lambda}(s)| + 1) |\Delta u_{\varepsilon,\lambda}(s)| dx \\ &\leq \frac{\varepsilon}{2} \int_{\Omega} |\Delta u_{\varepsilon,\lambda}(s)|^2 dx + \frac{c_3^2}{2} \int_{\Omega} (|u_{\varepsilon,\lambda}(s)| + 1)^2 dx \\ &\leq \frac{\varepsilon}{2} |\Delta u_{\varepsilon,\lambda}(s)|_H^2 + c_3^2 (|u_{\varepsilon,\lambda}(s)|_H^2 + |\Omega|) \end{aligned} \quad (6.3)$$

and

$$(\Delta f(s), u_{\varepsilon,\lambda}(s))_H \leq \frac{1}{2} |\Delta f(s)|_H^2 + \frac{1}{2} |u_{\varepsilon,\lambda}(s)|_H^2 \quad (6.4)$$

for a.a.  $s \in (0, T)$ . Therefore, by combining (6.1)–(6.4) and integrating over  $(0, t)$  with respect to  $s$ , we infer that

$$\begin{aligned} & \frac{1}{2} |u_{\varepsilon,\lambda}(t)|_H^2 + \frac{\lambda}{2} |\nabla u_{\varepsilon,\lambda}(t)|_{H^d}^2 + \frac{\varepsilon}{2} \int_0^t |\Delta u_{\varepsilon,\lambda}(s)|_H^2 ds \\ & \leq \frac{1}{2} |u_{0\varepsilon}|_H^2 + \frac{\lambda}{2} |\nabla u_{0\varepsilon}|_{H^d}^2 + c_3^2 \int_0^t (|u_{\varepsilon,\lambda}(s)|_H^2 + |\Omega|) ds \\ & \quad + \frac{1}{2} \int_0^t |\Delta f(s)|_H^2 ds + \frac{1}{2} \int_0^t |u_{\varepsilon,\lambda}(s)|_H^2 ds \end{aligned} \quad (6.5)$$

for all  $t \in [0, T]$ , thanks to the monotonicity of  $\beta_{\lambda}$ . At this point, we let  $\lambda \leq \varepsilon$  (which is always possible since  $\lambda$  is going to 0 before  $\varepsilon$ ) and recall (2.9) that entails

$$\frac{1}{2} |u_{0\varepsilon}|_H^2 + \frac{\lambda}{2} |\nabla u_{0\varepsilon}|_{H^d}^2 \leq c_4.$$

Then, we apply the Gronwall inequality and obtain

$$\begin{aligned} & |u_{\varepsilon,\lambda}(t)|_H^2 + \lambda |\nabla u_{\varepsilon,\lambda}(t)|_{H^d}^2 + \varepsilon \int_0^t |\Delta u_{\varepsilon,\lambda}(s)|_H^2 ds \\ & \leq \left\{ c_4 + |\Delta f|_{L^2(0,T;H)}^2 + 2c_3^2 |\Omega| T \right\} \exp \{ (2c_3^2 + 1) T \} \end{aligned} \quad (6.6)$$

for all  $t \in [0, T]$ . Hence, going back to the proof of Lemma 3.1, let us point out the inequalities (3.6) and (3.7): here we do not use (3.7) and from (3.6) we arrive at

$$\begin{aligned} & \frac{1}{2} \int_0^t |u'_{\varepsilon, \lambda}(s)|_{V^*}^2 ds + \lambda \int_0^t |u'_{\varepsilon, \lambda}(s)|_H^2 ds \\ & + \frac{\varepsilon}{2} |\nabla u_{\varepsilon, \lambda}(t)|_{H^d}^2 + |\widehat{\beta}_\lambda(u_{\varepsilon, \lambda}(t))|_{L^1(\Omega)} \leq \frac{3}{2} c_4 + \frac{c_1}{4} + \frac{c_1 c_4}{8} + \frac{1}{2} \|f\|_{L^2(0, T; V)}^2 \end{aligned} \quad (6.7)$$

for all  $t \in [0, T]$ , which replaces (3.10). Now, by adding (6.6) and (6.7) we obtain the useful bound to continue with other lemmas and end the proof of Theorem 2.3 also in this case, thus avoiding the assumption (A2).  $\square$

Moreover, in the framework of assumptions (A1), (A3) with (5.1), (A5) and (A6), we can also improve the error estimate stated in Theorem 5.1.

**Theorem 6.1.** *Assume (A1), (A3), (A5) and (A6) with (5.1) and (5.2). For  $\varepsilon \in (0, \bar{\varepsilon}]$ , let  $(u_\varepsilon, \mu_\varepsilon, \xi_\varepsilon)$  be a solution of problem  $(P)_\varepsilon$  and let  $(u, \mu, \xi)$  be a solution of the problem  $(P)$ . Then, there exists a constant  $C^* > 0$ , depending only on the data, such that*

$$|u_\varepsilon - u|_{C([0, T]; V^*)}^2 + \int_0^T (\xi_\varepsilon(s) - \xi(s), u_\varepsilon(s) - u(s))_H ds \leq C^* \varepsilon^{1/2} \quad (6.8)$$

for all  $\varepsilon \in (0, \bar{\varepsilon}]$ .

**Proof.** Compare (5.7) and (6.6): it turns out that the boundedness in  $L^2(0, T; H)$  of  $\{\varepsilon^{2/3} \Delta u_\varepsilon\}_{\varepsilon > 0}$  is improved to the one of  $\{\varepsilon^{1/2} \Delta u_\varepsilon\}_{\varepsilon > 0}$ . Therefore, in the proof of Theorem 5.1 the key estimate (5.9) can be replaced by

$$\begin{aligned} & \varepsilon \int_0^T (\Delta u_\varepsilon(s), u_\varepsilon(s) - u(s))_H ds \\ & \leq \varepsilon^{1/2} \left\{ \varepsilon \int_0^T |\Delta u_\varepsilon(s)|_H^2 ds \right\}^{1/2} \left\{ \int_0^T |u_\varepsilon(s) - u(s)|_H^2 ds \right\}^{1/2} \leq C_4^* \varepsilon^{1/2}, \end{aligned} \quad (6.9)$$

for some constant  $C_4^*$  depending only on  $c_1, c_3, c_4, \|\Delta f\|_{L^2(0, T; H)}, M_1, |\Omega|$  and  $T$ . Recalling now (5.2) and (5.10), it is straightforward to derive (6.8) from (5.8).  $\square$

The convergence result and the error estimate shown in this section turn out to be an improvement which does without the assumption (A2) and exploits (A6) instead. Actually, thanks to this, we can treat more general cases of  $\beta$ , in particular they apply to the problems described in Examples 5 and 6.

**Remark 6.2.** We note that in the framework of this section (cf. (A6)), the limit problem in (2.19)–(2.21) can be rewritten as

$$\langle u'(t), z \rangle_{V^*, V} + \int_\Omega \nabla \xi(t) \cdot \nabla z dx = \int_\Omega g(t) z dx \quad \text{for all } z \in V,$$

for a.a.  $t \in (0, T)$ , with

$$\xi \in \beta(u) \quad \text{a.e. in } Q, \quad u(0) = u_0 \quad \text{a.e. in } \Omega.$$

Here, the variable  $\mu$  disappears. We point out that now it is also  $\xi$ , and not only  $\xi - f$ , to formally satisfy the Neumann homogeneous boundary condition

$$\partial_\nu \xi = 0 \quad \text{a.e. on } \Sigma. \quad (6.10)$$

Owing to the zero mean value condition for  $g$ , now (6.10) is a necessary condition for the existence of solutions.

## 7 Appendix

We use the same notation as in the previous sections for function spaces.

**Lemma A.** *Let  $u_0 \in H$  with  $\widehat{\beta}(u_0) \in L^1(\Omega)$  and  $m_0 \in \text{int } D(\beta)$ . Then there exist a family  $\{u_{0\varepsilon}\}_{\varepsilon \in (0,1]} \subset V$  and a positive constant  $C$  such that*

$$\begin{aligned} m(u_{0\varepsilon}) = m_0, \quad |u_{0\varepsilon}|_H^2 \leq C, \quad \int_\Omega \widehat{\beta}(u_{0\varepsilon}) dx \leq C, \quad \varepsilon |\nabla u_{0\varepsilon}|_{H^d}^2 \leq C \quad \text{for all } \varepsilon \in (0, 1], \\ u_{0\varepsilon} \rightarrow u \quad \text{strongly in } H \text{ as } \varepsilon \searrow 0. \end{aligned}$$

**Proof.** For each  $\varepsilon \in (0, 1]$ , we can take  $u_{0\varepsilon} \in W$  as the solution of

$$\begin{cases} u_{0\varepsilon} - \varepsilon \Delta u_{0\varepsilon} = u_0 & \text{a.e. in } \Omega, \\ \partial_\nu u_{0\varepsilon} = 0 & \text{a.e. in } \Gamma. \end{cases}$$

Then,  $m(u_{0\varepsilon}) = m(u_0) = m_0$  and  $u_{0\varepsilon} \rightarrow u_0$  strongly in  $H$  as  $\varepsilon \searrow 0$ . Indeed, testing the first equation by  $u_{0\varepsilon}$  and using the Young inequality, we find

$$\int_\Omega |u_{0\varepsilon}|^2 dx + \varepsilon \int_\Omega |\nabla u_{0\varepsilon}|^2 dx \leq \frac{1}{2} \int_\Omega |u_0|^2 dx + \frac{1}{2} \int_\Omega |u_{0\varepsilon}|^2 dx, \quad (7.1)$$

whence  $\{u_{0\varepsilon}\}_{\varepsilon \in (0,1]}$  is bounded in  $H$  and  $\varepsilon u_{0\varepsilon} \rightarrow 0$  strongly in  $V$  as  $\varepsilon \searrow 0$ . Then, from the equation it turns out that  $u_{0\varepsilon} \rightarrow u_0$  weakly in  $H$ , when passing to the limit in

$$\int_\Omega (u_{0\varepsilon} - u_0) z dx + \varepsilon \int_\Omega \nabla u_{0\varepsilon} \cdot \nabla z dx = 0 \quad \text{for all } z \in V. \quad (7.2)$$

Moreover, from (7.1) it follows that

$$\limsup_{\varepsilon \searrow 0} \int_\Omega |u_{0\varepsilon}|^2 dx \leq \int_\Omega |u_0|^2 dx.$$



This ensures convergence of norms and finally strong convergence of  $u_{0\varepsilon}$  to  $u_0$  in  $H$ . Next, taking  $z := \beta_{\tilde{\varepsilon}}(u_{0\varepsilon})$ , where  $\beta_{\tilde{\varepsilon}}$  is the Yosida approximation of  $\beta$  (treated in Section 3) at  $\tilde{\varepsilon} \in (0, 1]$ , and using the definition of the subdifferential lead to

$$\begin{aligned} \int_{\Omega} (\hat{\beta}_{\tilde{\varepsilon}}(u_{0\varepsilon}) - \hat{\beta}_{\tilde{\varepsilon}}(u_0)) dx &\leq \int_{\Omega} (u_{0\varepsilon} - u_0) \beta_{\tilde{\varepsilon}}(u_{0\varepsilon}) dx \\ &= -\varepsilon \int_{\Omega} \beta'_{\tilde{\varepsilon}}(u_{0\varepsilon}) |\nabla u_{0\varepsilon}|^2 dx \leq 0. \end{aligned}$$

Therefore, we have that

$$\int_{\Omega} \hat{\beta}_{\tilde{\varepsilon}}(u_{0\varepsilon}) dx \leq \int_{\Omega} \hat{\beta}_{\tilde{\varepsilon}}(u_0) dx \leq \int_{\Omega} \hat{\beta}(u_0) dx \leq C;$$

if we take  $\tilde{\varepsilon} < \varepsilon$  and pass to the limit as  $\tilde{\varepsilon} \searrow 0$ , then we deduce that

$$\int_{\Omega} \hat{\beta}(u_{0\varepsilon}) dx = \lim_{\tilde{\varepsilon} \searrow 0} \int_{\Omega} \hat{\beta}_{\tilde{\varepsilon}}(u_{0\varepsilon}) dx \leq C.$$

Thus, (2.9) is completely proved. As a remark, the additional condition (5.2) is also guaranteed since we can take  $z := \mathcal{N}(u_{0\varepsilon} - u_0)$  in (7.2).  $\square$

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